MULTISCALE INFERENCE FOR INVERSE PROBLEMS VIA REGULARISED HYPOTHESIS TESTING

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Multiscale inference

- We are interested in **specific features** of an unknown function or density $u^\dagger$ such as its modes, convexity, monotonicity, or support.
- Many such features can be described by the application of a family of **bounded linear functionals** φ corresponding to different **locations** and **length scales**.

Examples

- **Support** of a non-negative function $u^\dagger$ can be described by **non-negative functions** φ with $\operatorname{supp}\varphi = [a,b]$ for a **family of intervals** $[a,b]$ of different size and location: **Intersection** of $\operatorname{supp} u^\dagger$ with $[a,b]$ corresponds to $\int \varphi(x)u^\dagger(x)\,\mathrm{d}x = \langle \varphi, u^\dagger\rangle_{L^2} > 0$.
- **Monotonicity** or **convexity** of a function can be described using its 1st or 2nd **derivative**.

- We perform statistical inference for linear functionals by means of **hypothesis testing**. Specifically, we test

$$H_0 : \langle \varphi, u^\dagger\rangle = 0 \quad \text{vs.} \quad H_1 : \langle \varphi, u^\dagger\rangle > 0.$$

Inverse problem set-up

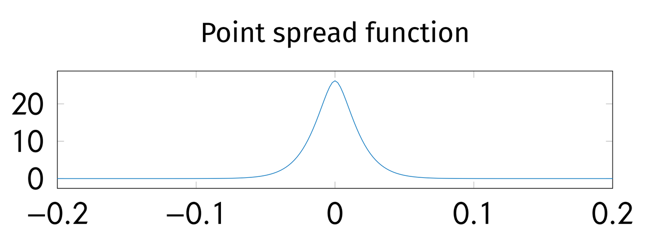
We are interested in a quantity $u^\dagger \in \mathcal{X}$ that **cannot be observed** directly. Instead, **indirect noisy measurements** Y are available. The relationship between $u^\dagger$ and Y is modelled as

$$Y = Tu^\dagger + \sigma Z,$$

where $T\colon \mathcal{X} \to \mathcal{Y}$ is a bounded **linear operator** between a real Banach space $\mathcal{X}$ and a real Hilbert space $\mathcal{Y}$, Z is a standard **white Gaussian noise process** on $\mathcal{Y}$, and $\sigma > 0$ is the noise level.

Examples

1. **Deconvolution:** Deblurring of an image, e.g., in medical imaging procedures, microscopy, astronomy. $T = T_{\mathrm{conv}}$ convolution operator on $\mathcal{X} = L^2(-1,1)$.



2. **Differentiation:** Estimating the second weak derivative of a function $y^\dagger \in H^2(0,1)$. $T = T_{\mathrm{antider}}$ antiderivative operator on $\mathcal{X} = L^2(0,1)$.
3. **Backwards heat equation:** Given a temperature distribution $y^\dagger$ on $[0,1]$ at time $t_0 > 0$, estimate the initial temperature distribution $u^\dagger$ at time 0. $T = T_{\mathrm{heat}}$ solution operator to forward heat equation on $\mathcal{X} = L^2(0,1)$.

Difficulty: The **ill-posed** of the problem prevents the reconstruction of $u^\dagger$ using standard methods.

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Regularised hypothesis testing [1]

- **Problem:** Classical plug-in tests

$$\Psi_0(Y) := \mathbb{1}_{\langle Y,\Phi_0\rangle > c} = \begin{cases} 1 & \text{if } \langle Y,\Phi_0\rangle > c,\\ 0 & \text{otherwise,}\end{cases}$$

  based upon **unbiased linear estimators** $\langle Y,\Phi_0\rangle$ for $\langle \varphi, u^\dagger\rangle$ can have **arbitrarily low power** or may **not** be **available** (when $\varphi \notin \operatorname{ran} T^*$) due to ill-posedness of the problem.
- **Idea:** Use plug-in tests $\Psi_\Phi(Y) := \mathbb{1}_{\langle Y,\Phi\rangle > c}$ based upon **linear estimators** $\langle Y,\Phi\rangle$ **related to variational regularisation methods** to overcome these issues, i.e., choose the probe element Φ as minimiser of an objective functional.
- Under certain regularity assumptions on $u^\dagger$, for **any** $\varphi \in \overline{\operatorname{ran} T^*}$, **any** nonzero $\Phi \in \mathcal{Y}$, and **any** $\alpha \in (0,1)$, the rejection threshold c can be chosen such that the **regularised test** Ψ_Φ has at most **level** α.

Possible choices of probe element Φ

1. Choose $\Phi \in \mathcal{Y}$ to **maximise** the **power** of the regularised test Ψ_Φ for testing H_0 against H_1 [1].
2. Choose $\Phi \in \mathcal{Y}$ as **Tikhonov-regularised solution** to $T^*\Phi_0 = \varphi$. Such a Tikhonov-regularised test Ψ_Φ corresponds to a **maximum a posteriori test** based upon a Gaussian prior distribution, which is constructed using a **Bayesian approach** [2].

Optimal regularised hypothesis testing [1]

Difficulties: The **objective functional** related to the power of the regularised test Ψ_Φ is **not convex** and **requires knowledge of the truth** $u^\dagger$.

Solutions

1. Show that the original optimisation problem is equivalent to a **constrained convex surrogate problem** that admits an efficient numerical solution.
2. **Estimate the power** of the regularised test Ψ_Φ based upon the data Y and **minimise** a corresponding **empirical objective functional**.

Restrictions: This approach requires **two independent measurements** Y_1 and Y_2.

Maximum a posteriori testing [2]

We consider the problem from a **Bayesian perspective** and model

$$Y = TU + \sigma Z,$$

where a **Gaussian prior distribution** $\mathcal{N}(m_0, C_0)$ is assigned to U, and $\mathcal{X}$ is a real Hilbert space. The **maximum a posteriori (MAP) test** Ψ_{MAP} rejects if

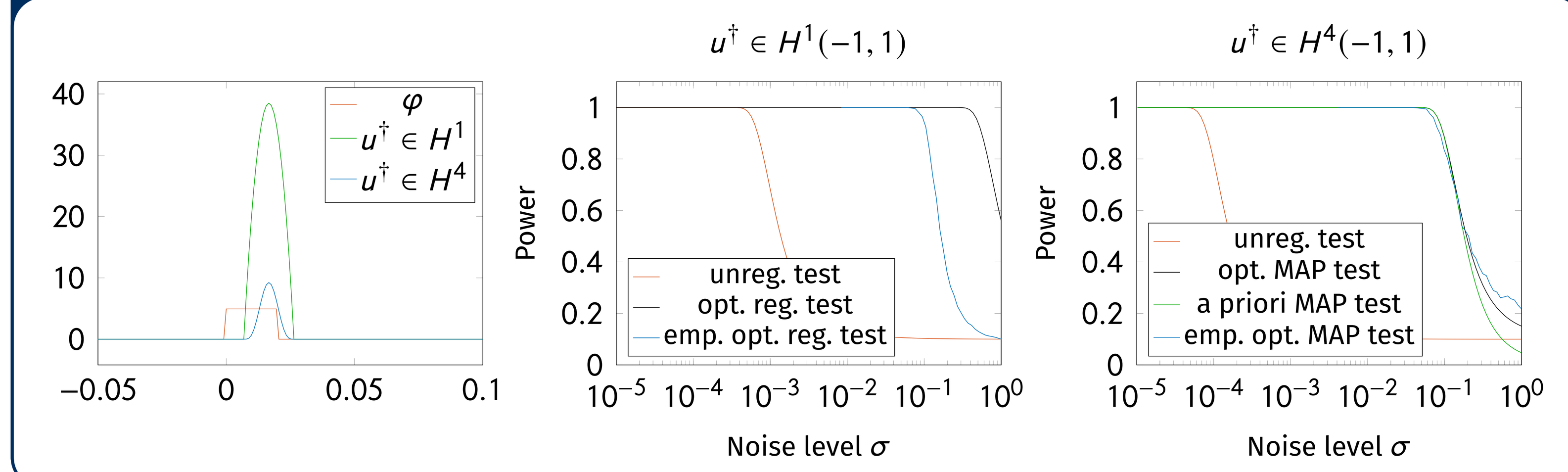
$$\mathbb{P}\left[\langle \varphi, U\rangle > 0 \,|\, Y\right] > \mathbb{P}\left[\langle \varphi, U\rangle \le 0 \,|\, Y\right].$$

It corresponds to a **regularised test** $\Psi_{\mathrm{MAP}} = \Psi_{\Phi_{\mathrm{MAP}}}$, where Φ_{MAP} is the **minimiser** of a **Tikhonov–Phillips functional**.

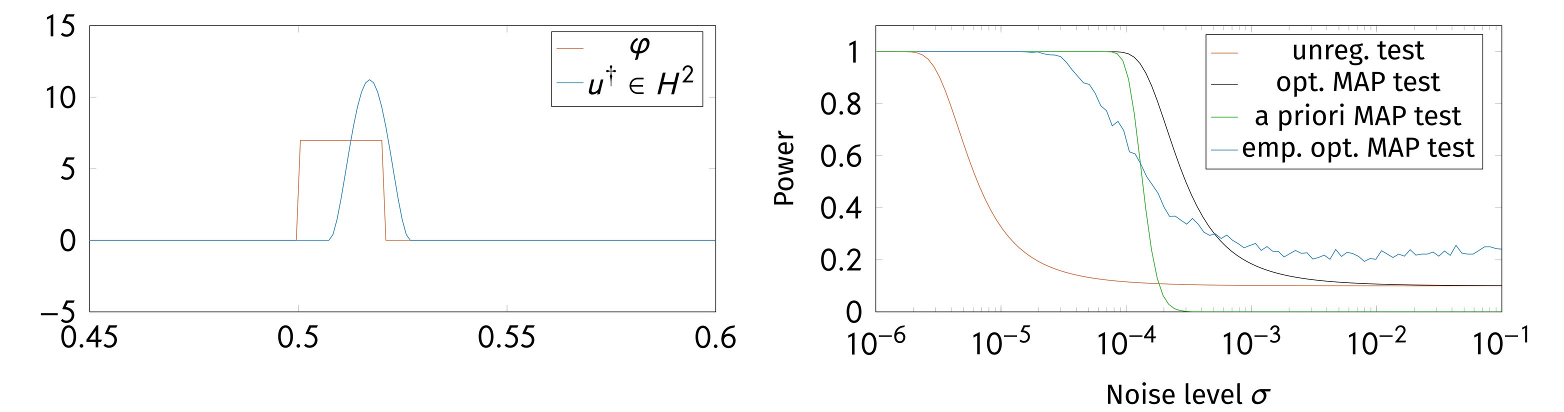
Numerical simulations [1, 2]

- Consider **indicator function** φ on the interval $[c, c+l]$ with $l \approx 0.020$.
- Choose **truth** $u^\dagger$ as symmetric β-kernel on the interval $[c+\frac{1}{3}l, c+\frac{4}{3}l]$.
- Construct MAP test using **prior covariance** $C_0 := \gamma^2 T^*T$ with $\gamma > 0$.
- Choose γ **a priori** depending on the noise level σ, or **maximise** the **estimated power** of the MAP test based upon the data Y.
- Implementation of forward operator T and computation of probe element Φ_{MAP} for MAP test via fast Fourier transform.
- Computation of probe element $\Phi^\dagger$ for optimal regularised test using primal dual proximal splitting method.

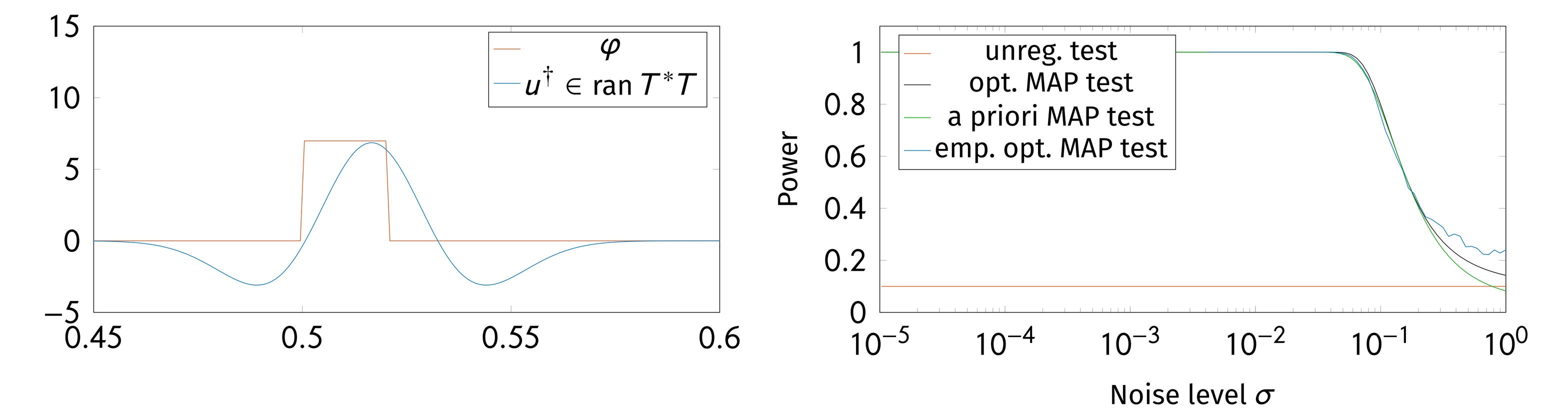
Deconvolution



Differentiation



Backwards heat equation



References

[1] R. Kretschmann, D. Wachsmuth, and F. Werner. Optimal regularized hypothesis testing in statistical inverse problems. *Inverse Problems*, 40(1):015013, 2024.

[2] R. Kretschmann and F. Werner. Maximum a posteriori testing in statistical inverse problems. *Inverse Problems and Imaging*, 19(6):1268–1301, 2025.